MULTIPLICITY PRESERVATION FOR ORTHOGONAL-SYMPLECTIC AND UNITARY DUAL PAIR CORRESPONDENCES

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Abstract. Over a non-archimedean local field of characteristic zero, we prove the multiplicity preservation for orthogonal-symplectic dual pair correspondences and unitary dual pair correspondences.

1. Introduction

Fix a non-archimedean local field k of characteristic zero, and a continuous involution τ on it. Denote by k_0 the fixed points of τ. Then either $\mathrm{k} = \mathrm{k}_0$ or k is a quadratic extension of k_0. Let $\epsilon = \pm 1$ and let E be an ϵ-hermitian space, namely it is a finite dimensional k-vector space, equipped with a non-degenerate k_0-bilinear map

$$\langle\, ,\,\rangle_E : E \times E \to \mathrm{k}$$

satisfying

$$\langle u, v\rangle_E = \epsilon \langle v, u\rangle_E^\tau, \quad \langle au, v\rangle_E = a\langle u, v\rangle_E, \quad a \in A,\ u, v \in E.$$

Write $\epsilon' = -\epsilon$, and let $(E', \langle\, ,\,\rangle_{E'})$ be an ϵ'-hermitian space. Then

$$\mathbf{E} := E \otimes_{\mathrm{k}} E'$$

is a k_0-symplectic space under the form

$$\langle u \otimes u', v \otimes v'\rangle_{\mathbf{E}} := \mathrm{tr}_{\mathrm{k}/\mathrm{k}_0}(\langle u, v\rangle_E\, \langle u', v'\rangle_{E'}).$$

Denote by

$$1 \to \{\pm 1\} \to \widetilde{\mathrm{Sp}}(\mathbf{E}) \to \mathrm{Sp}(\mathbf{E}) \to 1 \tag{1}$$

the metaplectic cover of the symplectic group $\mathrm{Sp}(\mathbf{E})$. Denote by

$$\mathbf{H} := \mathbf{E} \times \mathrm{k}_0$$

the Heisenberg group associated to $\mathbf{E}$, whose multiplication is given by

$$(u, t)(u', t') := (u + u', t + t' + \langle u, u'\rangle_{\mathbf{E}}).$$

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The group $\mathrm{Sp}(\mathbf{E})$ acts on $\mathbf{H}$ as automorphisms by

$$g.(u,t) := (gu,t). \tag{2}$$

It induces an action of $\widetilde{\mathrm{Sp}}(\mathbf{E})$ on $\mathbf{H}$, and further defines a semidirect product $\widetilde{\mathrm{Sp}}(\mathbf{E}) \ltimes \mathbf{H}$.

Fix a non-trivial character ψ of k, and denote by ω_ψ the corresponding smooth oscillator representation of $\widetilde{\mathrm{Sp}}(\mathbf{E}) \ltimes \mathbf{H}$. Up to isomorphism, this is the only genuine smooth representation which, as a representation of $\mathbf{H}$, is irreducible and has central character ψ. Recall that in general, if H is a group together with an embedding of $\{\pm 1\}$ in its center, a representation of H is called genuine if the element $-1 \in H$ acts via the scalar multiplication by -1.

Denote by G the group of all k-linear automorphisms of E which preserve the form $\langle\,,\,\rangle_E$. It is thus an orthogonal group, a symplectic group or a unitary group. The group G is obviously mapped into $\mathrm{Sp}(\mathbf{E})$. Define the fiber product

$$\widetilde{G} := \widetilde{\mathrm{Sp}}(\mathbf{E}) \times_{\mathrm{Sp}(\mathbf{E})} G,$$

which is a double cover of G. Similarly, we define G' and $\widetilde{G}'$. As usual, the product group $\widetilde{G} \times \tilde{G}'$ is mapped into $\widetilde{\mathrm{Sp}}(\mathbf{E})$.

The goal of this paper is to prove the following theorem, which is usually called the multiplicity preservation for theta correspondences, and is also called the Multiplicity One Conjecture by Rallis in [Ra84]. It is complementary to the famous Local Howe Duality Conjecture.

Theorem A. *For every genuine irreducible admissible smooth representation π of $\widetilde{G}$, and π' of $\widetilde{G}'$, one has that*

$$\dim \mathrm{Hom}_{\widetilde{G}\times\widetilde{G}'}(\omega_\psi, \pi\otimes\pi') \leq 1.$$

When the residue characteristic of k is odd, Theorem A is proved by Waldspurger in [Wa90]. The archimedean analog of Theorem A is proved by Howe in [Ho89].

2. A geometric result

We continue with the notation of the Introduction. Following [MVW87, Proposition 4.I.2], we extend G to a larger group $\breve{G}$, which contains G as a subgroup of index two, and consists pairs $(g,\delta) \in \mathrm{GL}_{\mathrm{k}_0}(E) \times \{\pm 1\}$ such that either

$$\delta = 1 \quad \text{and} \quad g \in G,$$

or

$$\begin{cases} \delta = -1, \\ g(au) = a^\tau g(u), \qquad a \in \mathrm{k},\ u \in E, \qquad \text{and} \\ \langle gu, gv\rangle_E = \langle v, u\rangle_E, \quad u, v \in E. \end{cases}$$

Similarly, we define a group $\breve{G}'$ and a group $\breve{\mathrm{Sp}}(\mathbf{E})$, which extend G' and $\mathrm{Sp}(\mathbf{E})$, respectively.

In general, if a group $\breve{H}$ is equipped with a subgroup H of index two, we will associate on it the nontrivial quadratic character which is trivial on H. We use χ_H to indicate this character.

Denote the fiber product

$$\breve{\mathbf{G}} := \breve{G} \times_{\{\pm 1\}} \breve{G}' = \{(g, g', \delta) \mid (g, \delta) \in \breve{G},\ (g', \delta) \in \breve{G}'\},$$

which contains

$$\mathbf{G} := G \times G'$$

as a subgroup of index two. Define a group homomorphism

$$\begin{aligned} \xi : \breve{\mathbf{G}} &\to \breve{\mathrm{Sp}}(\mathbf{E}), \\ (g, g', \delta) &\mapsto (g \otimes g', \delta). \end{aligned} \tag{3}$$

Let $\breve{\mathrm{Sp}}(\mathbf{E})$ act on the Heisenberg group $\mathbf{H}$ as group automorphisms by

$$(g, \delta).(u, t) := (gu, \delta t), \tag{4}$$

which extends the action (2). By using the homomorphism ξ, this induces an action of $\breve{\mathbf{G}}$ on $\mathbf{H}$, and further defines a semidirect product

$$\breve{\mathbf{J}} := \breve{\mathbf{G}} \ltimes \mathbf{H},$$

which contains

$$\mathbf{J} := \mathbf{G} \ltimes \mathbf{H}$$

as a subgroup of index two.

Let the group

$$\{\pm 1\} \ltimes (\breve{\mathbf{G}} \times \breve{\mathbf{G}}) \tag{5}$$

act on $\breve{\mathbf{J}}$ by

$$(\delta, \breve{\mathbf{g}}_1, \breve{\mathbf{g}}_2).\breve{\mathbf{j}} := (\breve{\mathbf{g}}_1 \breve{\mathbf{j}} \breve{\mathbf{g}}_2^{-1})^\delta, \tag{6}$$

where the semidirect product in (5) is defined by the action

$$-1.(\breve{\mathbf{g}}_1, \breve{\mathbf{g}}_2) := (\breve{\mathbf{g}}_2, \breve{\mathbf{g}}_1).$$

The fibre product

$$\{\pm 1\} \ltimes_{\{\pm 1\}} (\breve{\mathbf{G}} \times_{\{\pm 1\}} \breve{\mathbf{G}}) = \{(\delta, \breve{\mathbf{g}}_1, \breve{\mathbf{g}}_2) \mid \chi_{\mathbf{G}}(\breve{\mathbf{g}}_1) = \chi_{\mathbf{G}}(\breve{\mathbf{g}}_2) = \delta\}$$

is a subgroup of (5). It contains $\mathbf{G} \times \mathbf{G}$ as a subgroup of index two, and stabilizes $\mathbf{J}$ under the action (6).

We prove the following proposition in the remaining of this section.

Proposition 2.1. *Every $\mathbf{G} \times \mathbf{G}$-orbit in $\mathbf{J}$ is stable under the group $\{\pm 1\} \ltimes_{\{\pm 1\}} (\breve{\mathbf{G}} \times_{\{\pm 1\}} \breve{\mathbf{G}})$.*

Let $\breve{\mathbf{G}}$ act k_0-linearly on $\mathbf{E}$ by

$$(g, g', \delta).u \otimes u' := \delta g u \otimes g' u'. \tag{7}$$

Lemma 2.2. *Every $\mathbf{G}$-orbit in $\mathbf{E}$ is $\breve{\mathbf{G}}$-stable.*

We first prove

Lemma 2.3. *Lemma 2.2 implies Proposition 2.1.*

Proof. Note that every $\mathbf{G} \times \mathbf{G}$-orbit in $\mathbf{J}$ intersect the subgroup $\mathbf{H}$, and the subgroup

$$\{\pm 1\} \times_{\{\pm 1\}} (\Delta(\breve{\mathbf{G}})) \quad \text{of} \quad \{\pm 1\} \ltimes_{\{\pm 1\}} (\breve{\mathbf{G}} \times_{\{\pm 1\}} \breve{\mathbf{G}})$$

stabilizes $\mathbf{H}$, where "Δ" stands for the diagonal group. Therefore in order to prove Proposition 2.1, it suffices to show that every $\Delta(\mathbf{G})$-orbit in $\mathbf{H}$ is $\{\pm 1\} \times_{\{\pm 1\}} (\Delta(\breve{\mathbf{G}}))$-stable. Identify $\{\pm 1\} \times_{\{\pm 1\}} (\Delta(\breve{\mathbf{G}}))$ with $\breve{\mathbf{G}}$. Then as a $\breve{\mathbf{G}}$-space,

$$\mathbf{H} = \mathbf{E} \times \mathrm{k},$$

where $\mathbf{E}$ carries the action (7), and k carries the trivial $\breve{\mathbf{G}}$-action. This finishes the proof. $\square$

Let $\breve{\mathbf{G}}$ act k_0-linearly on

$$\mathbf{E}' := \mathrm{Hom}_{\mathrm{k}}(E, E')$$

by

$$((g, g', \delta).\phi)(u) := \delta\, g'(\phi(g^\tau u)),$$

where

$$g^\tau := \begin{cases} g^{-1}, & \text{if } \delta = 1, \\ \epsilon g^{-1}, & \text{if } \delta = -1. \end{cases}$$

Then one checks that the k_0-linear isomorphism

$$\begin{array}{rcl} \mathbf{E} & \to & \mathbf{E}', \\ u \otimes u' & \mapsto & (v \mapsto \langle v, u \rangle_E u') \end{array}$$

is $\breve{\mathbf{G}}$-intertwining. Therefore Lemma 2.2 is equivalent to the following

Lemma 2.4. *Every $\mathbf{G}$-orbit in $\mathbf{E}'$ is $\breve{\mathbf{G}}$-stable.*

Denote by
$$\mathfrak{g} := \{x \in \mathrm{End}_{\mathrm{k}}(E) \mid \langle xu, v\rangle_E + \langle u, xv\rangle_E = 0\}$$
the Lie algebra of G, and put
$$\tilde{\mathfrak{g}} := \{(x, F) \mid x \in \mathfrak{g}, F \text{ is a k-subspace of } E,\ x|_F = 0\}.$$
Let $\breve{G}$ act on $\tilde{\mathfrak{g}}$ by
$$(g, \delta).(x, F) := (\delta g x g^{-1}, gF).$$
The action of $\breve{\mathbf{G}}$ on $\mathbf{E}'$ induces an action of
$$\breve{G} = \breve{\mathbf{G}}/G'$$
on the quotient space $G'\backslash\mathbf{E}'$.

Lemma 2.5. *There is a $\breve{G}$-intertwining embedding from $G'\backslash\mathbf{E}'$ into $\tilde{\mathfrak{g}}$.*

Proof. Recall that the map
$$x \mapsto \langle x\,\cdot, \,\cdot\rangle_E$$
establishes a k_0-linear isomorphism form $\mathfrak{g}$ onto the space of ϵ'-hermitian forms on the k-vector space E. Define a map
$$\begin{array}{rcl} \Xi : \mathbf{E}' = \mathrm{Hom}_{\mathrm{k}}(E, E') & \rightarrow & \tilde{\mathfrak{g}}, \\ \phi & \mapsto & (x, F), \end{array}$$
where F is the kernel of ϕ, and x is specified by the formula
$$\langle \phi(u), \phi(v)\rangle_{E'} = \langle xu, v\rangle_E, \quad u, v \in E.$$
Use Witt's Theorem, one finds that two elements of $\mathbf{E}'$ stay in the same G'-orbit precisely when they have the same image under the map Ξ. Therefore Ξ reduces to an embedding
$$G'\backslash\mathbf{E}' \hookrightarrow \tilde{\mathfrak{g}},$$
which is checked to be $\breve{G}$-intertwining. □

The following lemma is stated in [MVW87, Proposition 4.I.2]. We omit its proof.

Lemma 2.6. *For every $(x, F) \in \tilde{\mathfrak{g}}$, there is an element $(g, -1) \in \breve{G}$ such that*
$$gxg^{-1} = -x \quad \text{and} \quad gF = F.$$

In other words, every element of $\tilde{\mathfrak{g}}$ is fixed by an element of $\breve{G} \setminus G$. Therefore every G-orbit in $\tilde{\mathfrak{g}}$ is $\breve{G}$-stable. Now Lemma 2.5 implies that every G-orbit in $G'\backslash\mathbf{E}'$ is $\breve{G}$-stable, or equivalently, every $\mathbf{G}$-orbit in $\mathbf{E}'$ is $\breve{\mathbf{G}}$-stable. This proves Lemma 2.4, and the proof of Proposition 2.1 is now complete.

3. Proof of Theorem A

We first recall the notions of distributions and generalized functions on a t.d. group, i.e., a topological group whose underlying topological space is Hausdorff, secondly countable, locally compact and totally disconnected. Let H be a t.d. group. A distribution on H is defined to be a linear functional on $C_0^\infty(H)$, the space of compactly supported, locally constant (complex valued) functions on H. Denote by $D_0^\infty(H)$ the space of compactly supported distributions on H which are locally scalar multiples of a fixed haar measure. A generalized function on H is defined to be a linear functional on $D_0^\infty(H)$.

Recall the following version of Gelfand-Kazhdan criteria.

Lemma 3.1. *Let S be a closed subgroup of a t.d. group H, and let σ be a continuous anti-automorphism of H. Assume that every bi-S-invariant generalized function on H is σ-invariant. Then for every irreducible admissible smooth representations π of H, one has that*

$$\dim \operatorname{Hom}_S(\pi, \mathbb{C}) \cdot \dim \operatorname{Hom}_S(\pi^\vee, \mathbb{C}) \leq 1.$$

Here and henceforth, we use "$^\vee$" to indicate the contragredient of an admissible smooth representation. Lemma 3.1 is proved in a more general form in [SZ08, Theorem 2.2] for real reductive groups. The same proof works here and we omit the details.

Now we continue with the notation of the last section.

Lemma 3.2. *If a generalized function on $\mathbf{J}$ is $\mathbf{G} \times \mathbf{G}$ invariant, then it is also invariant under the group $\{\pm 1\} \ltimes_{\{\pm 1\}} (\breve{\mathbf{G}} \times_{\{\pm 1\}} \breve{\mathbf{G}})$.*

Proof. Note that the t.d. group $\breve{\mathbf{J}}$ is unimodular. Therefore we may replace "generalized function" by "distribution" in the proof of the lemma. Then by [BZ76, Theorem 6.9 and Theorem 6.15 A], the lemma is implied by Proposition 2.1. □

Lemma 3.3. *For every irreducible admissible smooth representations Π of $\mathbf{J}$, one has that*

$$\dim \operatorname{Hom}_{\mathbf{G}}(\Pi, \mathbb{C}) \cdot \dim \operatorname{Hom}_{\mathbf{G}}(\Pi^\vee, \mathbb{C}) \leq 1.$$

Proof. The lemma follows from Lemma 3.1 and Lemma 3.2 by noting that an element of the form

$$(-1, \breve{\mathbf{g}}, \breve{\mathbf{g}}) \in \{\pm 1\} \ltimes_{\{\pm 1\}} (\breve{\mathbf{G}} \times_{\{\pm 1\}} \breve{\mathbf{G}})$$

acts as an anti-automorphism on $\mathbf{J}$. □

Let ω_ψ, π and π' be as in Theorem A. As in the proof of [Su08, Lemma 5.3], $\omega_\psi \otimes \pi^\vee \otimes \pi'^\vee$ is an irreducible admissible smooth representation of $\mathbf{J}$. Therefore

Lemma 3.3 implies that

$$\dim \mathrm{Hom}_{\mathbf{G}}(\omega_\psi \otimes \pi^\vee \otimes \pi'^\vee, \mathbb{C}) \cdot \dim \mathrm{Hom}_{\mathbf{G}}(\omega_\psi^\vee \otimes \pi \otimes \pi', \mathbb{C}) \leq 1. \tag{8}$$

By [Su09, Theorem 1.4], the two factors in the left hand side of (8) are equal. Therefore

$$\dim \mathrm{Hom}_{\mathbf{G}}(\omega_\psi \otimes \pi^\vee \otimes \pi'^\vee, \mathbb{C}) \leq 1,$$

and consequently,

$$\dim \mathrm{Hom}_{\mathbf{G}}(\omega_\psi, \pi \otimes \pi') \leq 1.$$

This finishes the proof of Theorem A.

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